

# On a Bicriterion Server Allocation Problem in a Multidimensional Erlang Loss System

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**Abstract.** In this work an optimization problem on a classical elementary stochastic system, modeled as an Erlang-B (M/M/x) loss system, is formulated by using a bicriteria approach. The problem is focused on the allocation of a given total of  $\kappa$  servers to a number of groups of servers capable of carrying certain offered traffic processes assumed as Poissonian in nature. Two main objectives are present in this formulation. Firstly a criterion of equity in the grade of service, measured by the call blocking probabilities, entails that the absolute difference between the blocking probabilities experienced by the calls in the different service groups must be as small as possible. Secondly a criterion of system economic performance optimization requires the total traffic carried by the system, to be maximized. Relevant mathematical results characterizing the two objective functions and the set  $\mathcal{N}$  of the non-dominated solutions, are presented. An algorithm for traveling on  $\mathcal{N}$  based on the resolution of single criterion convex problems, using a Newton-Raphson method, is also proposed. In each iteration the two first derivatives of the Erlang-B function in the number of circuits (a difficult numerical problem) are calculated using a method earlier proposed. Some computational results are also presented.

**Key words.** Multiobjective Convex Optimization, Communication Networks, Stochastic Models, Erlang Loss System.

**AMS subject classifications.** 90C29, 90B18, 90B22, 65K05, 33-04.

## 1 Introduction and Motivation

There are numerous problems involving stochastic systems where there is the contention of a number of  $S$  random demand processes for  $\kappa$  identical resources. This problem has received much attention in the literature on allocation of transport vehicles and of urban emergency units, such as police cars, fire engines and ambulances, and may be considered as a *server allocation problem*. Other applications of this class of problems have been studied recently.

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One example is found on parallel processing in computer systems, where different users classes generate jobs submitted to a computer system which is composed of  $S$  identical computer processors or peripherals. Other examples are allocation of transmission bandwidth in satellite communication systems, and dynamic shared memory in computer systems.

Several models with mathematical formulations of this class of problems, are found in the literature. Most of the models consider non-linear functions describing the behavior of the systems by using queueing theory. The question of optimizing their performance was considered on a single criterion basis. The used criterion is a parameter of grade of service or, alternatively, the system throughput which is a measure of the total carried traffic by the system. The extremely rapid evolution of telecommunication networks in terms of technologies, traffic growth and provided services has led to the emergence of significant number of new problems of network planning and design often involving multiple and conflicting factors. Many of these problems lead to the necessity of formulating mathematical models for decision support, including various criteria, often conflicting in nature. Therefore it can be affirmed that in many situations the mathematical models for decision support become more realistic and “powerful” (concerning practical applications) if the more relevant criteria are represented explicitly rather than aggregated *a priori* in a single function to be optimized. These factors led to the increasing interest in the development of multicriteria models and, in particular, multiobjective optimization models, in this broad area of network planning and design. Note that multicriteria models enable the relevant aspects of the decision problem to be explicitly represented in the mathematical formulation and the compromises to be made among the chosen objective to be treated in a mathematically consistent manner.

Remember that in the context of various conflicting criteria (corresponding to objective functions) the concept of optimal solution is replaced by the concept of non-dominated solution set (corresponding to the concept of efficient Pareto solution set in the objective function space). A non-dominated solution is a solution for which no improvement is possible in one objective function without worsening at least the value of one of the other objective functions.

An overview of areas of application of multicriteria analysis tools in communication network planning problems can be seen in [16]. An in-depth analysis of conceptual issues associated with the use of multicriteria analysis in telecommunication network design, in the framework of knowledge theory models was presented in [28]. A comprehensive review on multicriteria models dedicated to communication network planning and design problems can be seen in [5] and an overview on multicriteria formulations for routing problems in communication networks, including a case study on multiobjective routing optimization, is presented in [6].

In communication networks with traffic of telephone type, generated by a very large number of subscribers (as compared to the number of available transmission channels), the offered traffic is a stochastic point process that may be modeled as an homogeneous Poisson process with a certain intensity  $\lambda$  and this corresponds to assume that interarrival times are negative exponentially distributed with mean  $\lambda^{-1}$ . If we further assume a negative exponential distribution of the call service times, we are led to the well-known classical Erlang-B stochastic service

system model,  $M/M/x$  ( $x$  being the number of channels) originally proposed by A. K. Erlang in 1917 [8, 9].

The Erlang B and C formulas are true probability classics. Indeed, much of the theory was developed by A. K. Erlang [8, 9] and his colleagues prior to 1925 [3]. The subject has been extensively studied and applied by telecommunications engineers and mathematicians ever since. A nice introductory account, including some of the telecommunication subtleties, is provided by [7].

The Erlang B (or loss) formula gives the (steady-state) blocking probability in the Erlang loss model, *i.e.*, in the  $M/M/x$  model. This model has  $x$  identical servers and no queue. Customers arriving when all  $x$  servers are busy are blocked (lost) without affecting future arrivals. This model has a Poisson arrival process and IID (independent and identically distributed) service times, with an exponential distribution having finite mean (the two M's in  $M/M/x$  are for "Markov," referring to the "lack-of-memory" property of the exponential distribution).

In this work a basic optimization problem on a teletraffic system, modeled as an Erlang-B ( $M/M/x$ ) loss system, is formulated by using a bicriteria approach. The problem is focused on the allocation of a given total of  $\kappa$  transmission channels to a number of service groups, capable of carrying certain offered traffic processes assumed as Poissonian in nature and characterized by their means expressed in Erlang. Such system may be considered as particular generalization of the classical one-dimensional Erlang-B system. The available capacity of the system  $\kappa$ , could be partitioned into  $S = n + 1$  separate groups of servers (or channels in a telecommunication network) with dimensions  $x_1, x_2, \dots, x_{n+1}$  with  $\sum x_i = \kappa$ , such that each group is dedicated for exclusive use by the corresponding offered traffic.

Two main objectives are present in this formulation. Firstly a criterion of grade of service equity (or "fairness") entails that call blocking probabilities in all groups must be as small as possible leading to a first objective function (to be minimized) that is the maximal blocking probability experienced by source demands (or "calls" in telecommunication networks) offered to the different groups of servers. On the other hand a criterion (efficiency function) of global system performance optimization requires the total traffic carried by the system, to be maximized. The first objective may be considered as a stochastic formulation of a particular application of the Max-Min fairness assignment principle (MMF) proposed in [2]. A comprehensive analysis of the application of the MMF principle to various problems of communication network design can be seen in [24, Chap. 8]. A lexicographic optimization approach for solving MMF problems in telecommunication network design, is described in [29].

Some mathematical results characterizing the set  $\mathcal{N}$  of the non-dominated solutions, are presented in this paper. An algorithm for traveling on  $\mathcal{N}$  based on the resolution of single criterion convex problems, using Newton-Raphson method, is proposed. In each iteration the first two derivatives of the Erlang-B function in the number of servers (a difficult numerical problem) are calculated by using a method proposed by the authors [12]. Some numerical and graphical results are also presented.

The major contribution of the paper is the presentation of a bicriteria formulation for a

basic problem of stochastic optimization in teletraffic systems and the proposal of an exact method for its resolution. Other contributions are the derivation of important mathematical properties of the objective functions of the problem, namely the efficiency function and the equity function (formulated as a Max-Min Fairness Principle) and the presentation of methods for numerical calculation of the optimal solutions of the two objective functions as well as the proposal of an algorithm for traveling on the set of Pareto efficient solutions. This algorithm is based on the resolution of a sequence of single criterion convex programming problems, using a Newton-Raphson method.

This paper is organized as follows. The next section presents the assumptions of the mathematical model and the formulation of the Erlang-B bicriterion server allocation optimization problem. The mathematical properties of the two objective function of the formulated allocation problem are derived in Section 3. Also in this section, methods for numerical calculation of the optimal solutions for the two objective functions, using a Newton-Raphson approach, are presented. Section 4 analyses, in the form of three Lemmas, the properties of the conflict between the efficiency and equity objective functions. The resolution approach, including its mathematical foundations, namely the characterization of the Pareto solutions, the formulation of an auxiliary parametric single criterion constrained optimization problem are shown in Section 5 together with an algorithm for traveling on the set of non-dominated solutions. Some computational results for illustrating the effectiveness of the proposed algorithm are shown in Section 6 and some conclusions are drawn in the final section.

## 2 Description of the Model

### 2.1 Assumptions

Let us consider the classical stochastic Erlang-B loss system  $M/M/x$ , with Poisson arrival intensity  $\lambda$  and mean occupation time  $h = \mu^{-1}$  in any of the  $x$  servers. Then the mean number of arrivals during  $h$ ,  $a$ , defines the mean of the traffic offered, usually designated in teletraffic theory as *traffic offered* expressed in Erlang:

$$a = \frac{\lambda}{\mu}. \quad (1)$$

Let us now consider Figure 1 representing the Erlang-B system with  $x$  servers (usually designated as circuits or channels in teletraffic theory) with offered traffic  $a \in \mathbb{R}_+$ . All free channels are fully available to incoming calls. There is no waiting room, that is a call which finds all the servers occupied, abandons the system. The grade of service provided to the customers is usually measured in terms of the call congestion, that is, the probability that an arriving call finds all servers busy. If the system is in statistical equilibrium this blocking probability  $B(a, x)$  is given by the very well known Erlang-B formula [9]:

$$B(a, x) = \frac{a^x/x!}{\sum_{j=0}^x a^j/j!}. \quad (2)$$

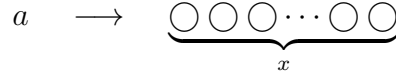


Figure 1: The Erlang Loss System

The Erlang-B formula plays an important role in many problems of teletraffic theory and this is probably the reason why it has been the subject of intensive study, as shown in references [17] [18] [19] [20] [26]. In several teletraffic studies the need arose to extend the definition of Erlang-B function to non-integer values of  $x$  by using its analytical extension, ascribed to R. Fortet [27, pag. 602]:

$$B(a, x) = \left( a \int_0^{+\infty} e^{-az} (1+z)^x dz \right)^{-1}. \quad (3)$$

This extension enables the number of circuits  $x$  to be considered as a nonnegative real value and is very important for defining approximation techniques and efficient optimization algorithms for teletraffic network dimensioning. The function defined by (3) (sometimes called continued Erlang-B function) is an higher transcendental function which may be related to the incomplete gamma function and to the confluent hypergeometric functions [18].

A brief review of the mathematical properties of the function  $B(a, x)$  defined by (3), will be given here having in mind its importance in the context of the developed analysis.

Initially, note that  $B(a, x) \in [0, 1]$ ,  $B(a, 0) = 1$ ,  $\forall a \in \mathbb{R}_+$ , and

$$\lim_{x \rightarrow \infty} B(a, x) = 0, \quad \forall a \in \mathbb{R}_+.$$

Various numerical procedures have been proposed for calculating  $B(a, x)$  — see for example [15] [18] [19] [21] [23]. The first order partial derivatives of  $B(a, x)$  with respect to  $a$  and  $x$  are given by (see for example [18]):

$$B'_a(a, x) = \frac{\partial B}{\partial a}(a, x) = \left[ \frac{x}{a} - 1 + B(a, x) \right] B(a, x), \quad (4)$$

$$B'_x(a, x) = \frac{\partial B}{\partial x}(a, x) = -[B(a, x)]^2 a \int_0^{+\infty} e^{-az} (1+z)^x \ln(1+z) dz. \quad (5)$$

From expression (5),  $B(a, x)$  is a strictly decreasing function of  $x$  for all  $a \in \mathbb{R}_+$ . Also, if  $x > 0$  then  $B'_a$  is always a positive value (see for example [19]).

Differentiation of (5) leads to the second order derivative of  $B(a, x)$  with respect to  $x$ :

$$\begin{aligned} B''_x(a, x) &= -2 B(a, x) B'_x(a, x) a \int_0^{+\infty} e^{-az} (1+z)^x \ln(1+z) dz - \\ &\quad - [B(a, x)]^2 a \int_0^{+\infty} e^{-az} (1+z)^x [\ln(1+z)]^2 dz. \end{aligned} \quad (6)$$

The frequently conjectured convexity of Erlang-B function (with respect to the variable  $x$ ) was proved by A. A. Jagers e E. A. Van Doorn [20]. In [14, 11, 4] it is shown that  $B''_x(a, x)$  is strictly

positive if  $x \geq 0$  for every  $a > 0$ . This result implies that  $B(a, x)$  is a strictly convex function of  $x$  if  $x \geq 0$ . The numerical calculus of  $B(a, x)$ ,  $B'_x(a, x)$  and  $B''_x(a, x)$  is very important for some algorithms presented in this paper. The method used for this purpose was proposed by the authors in a previous work [12, 13].

Since  $B(a, x)$  gives the proportion of lost calls, the function:

$$A_c(a, x) = a [1 - B(a, x)], \quad (7)$$

is normally designated by *carried traffic* and gives the average number of calls simultaneously in progress in the Erlang-B group.

The following function,

$$A_t(a, x) = a B(a, x), \quad (8)$$

is designated as *lost traffic* and sometimes is also designated as overflow traffic, whenever it is offered to another, second-choice system and gives the mean number of blocked calls during the mean service time. The relations between the functions introduced above are easily

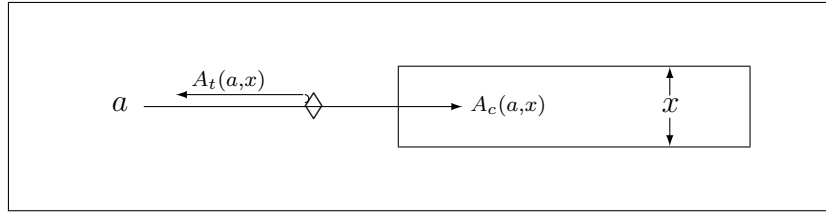


Figure 2: Flow model of a Erlang-B group in statistical equilibrium.

understood if we define an “analogous” deterministic flow model. Figure 2 shows such model, where we have a deterministic commodity flow with volume  $a$  offered to a transmission system (capacitated arc) of finite-capacity  $x$  through an access node. This node rejects a flow amount given by  $A_t(a, x)$  and accepts the remaining  $A_c(a, x)$  which effectively is transferred through (carried) by the arc. Note that this model is completely deterministic while an Erlang-B group is a stochastic system. Nevertheless, if we are interested only in the mean value of the variables which describe the behavior of the system in statistical equilibrium, this flow model is adequate.

The system represented by Figure 3 will be designated as  $\mathcal{E}_B^{n+1}$  system, and is composed of a sequence of Erlang-B groups with nonnegative capacities  $x_i$ ,  $i = 1(1)n + 1$ . The group  $i$  of the  $\mathcal{E}_B^{n+1}$  system has offered traffic  $a_i \in \mathbb{R}_+$ . The system is characterized by a parameter  $\kappa$  representing the total available capacity to be partitioned into  $n + 1$  slices, each of one is the capacity allocated to a separate Erlang-B group. It is easy to see that we have only  $n$  decision variables corresponding to the position of the separating lines  $L_i$  in Figure 3.

Two objectives are considered. The first criterion to be optimized is the system performance: the mean of the total number of calls in progress, that is  $\sum_{j=1}^{n+1} A_c(a_j, x_j)$ . This is a classical economic related criterion normally used in this area taking into account that the expectation

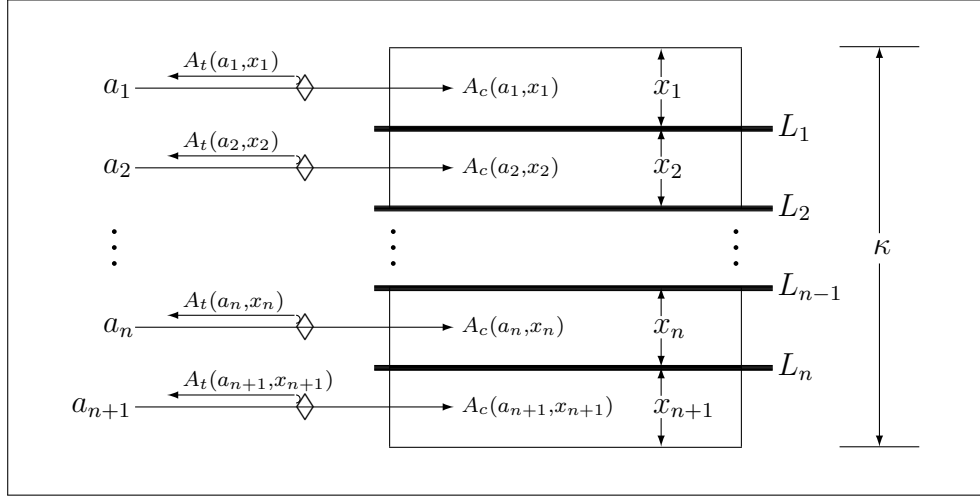


Figure 3: Resource allocation between  $n + 1$  Erlang-B groups: The available capacity of the system  $\kappa$ , could be partitioned into  $n + 1$  separate server groups of capacity  $x_1, x_2, \dots, x_{n+1}$ . The separating lines  $L_i$ ,  $i = 1(1)n$  may be vertically moved, changing the capacities  $x_i$  while the total capacity allocated  $\sum_{j=1}^{n+1} x_j = \kappa$  remains constant.

of the revenue of the telecommunication operator is proportional to the carried traffic. On the other hand, call blocking probabilities experienced by the calls in all the different service groups, must be as small as possible, leading to a first objective function (to be minimized) that is the maximal blocking probability experienced by source demands (or calls in telecommunication networks) offered to the different groups of servers. It will be seen that this criterion is an equity criterion too, since its optimal solution seeks to equalize the grade of service (measured by the blocking probabilities) among all service groups.

In this paper, a bicriterion formulation of the problem of allocating capacity to Erlang-B server groups, will be presented, together with algorithms for the numerical calculation of Pareto optimal solutions. Those algorithms may be used as a bicriterion decision aid tool on projects involving the  $\mathcal{E}_B^{n+1}$  system model.

## 2.2 The Erlang-B Bicriterion Allocation Problem

Firstly let us introduce the notation. Throughout this paper,  $\mathbb{R}_+^p$  and  $\mathbb{R}_\oplus^p$  will be used for designating the positive and nonnegative orthant of  $\mathbb{R}^p$ , respectively. If  $\mathbb{D}$  is a subset of  $\mathbb{R}^p$ , then the interior and the boundary of  $\mathbb{D}$  are denoted, respectively, by  $\mathbb{D}^\circ$  and  $\partial\mathbb{D}$ .

In connection with the model of the  $\mathcal{E}_B^{n+1}$  system, described in the previous section,  $\mathbf{a} \in \mathbb{R}_+^{n+1}$  is designated as offered traffic vector, and  $\bar{\mathbf{x}} \in \mathbb{R}_\oplus^{n+1}$  is the group capacity vector:

$$\begin{aligned} \mathbf{a} &= [a_1, a_2, \dots, a_n, a_{n+1}]^T, \\ \bar{\mathbf{x}} &= [x_1, x_2, \dots, x_n, x_{n+1}]^T, \end{aligned} \quad (9)$$

where  $a_j$  is the traffic offered to group  $j$  which has capacity  $x_j$  for  $j = 1(1)n + 1$ . Additionally,

let us suppose that the components of vector  $\mathbf{a}$  are ordered according to

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1}. \quad (10)$$

We are dealing with a problem with  $n$  decision variables such that a decision vector  $\mathbf{x}$  of  $\mathbb{R}_+^n$  is defined by the  $n$  first coordinates of the vector  $\bar{\mathbf{x}}$ , that is  $x_j$ ,  $j = 1(1)n$ .

Throughout the paper,  $b_j$  may replace  $B(a_j, x_j)$  whenever  $x_j$  (the capacity of group  $j$ ) may be implied from the context. In that case,  $b'_j$  and  $b''_j$  may replace, in similar form,  $B'_x(a_j, x_j)$  and  $B''_x(a_j, x_j)$ , respectively.

Therefore, the bicriterion resource allocation problem may be stated as follows.

**Problem 1 (Erlang-B Bicriterion Resource Allocation)**

Given  $n \in \mathbb{N}$ ,  $\kappa \in \mathbb{R}_+$  and an offered traffic vector  $\mathbf{a} \in \mathbb{R}_+^{n+1}$ , calculate the set of Pareto optimal solutions of:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_1(\mathbf{x}) = \sum_{j=1}^n a_j B(a_j, x_j) + a_{n+1} B\left(a_{n+1}, \kappa - \sum_{j=1}^n x_j\right) \\ \min_{\mathbf{x}} \quad & f_2(\mathbf{x}) = \max \left\{ B(a_1, x_1), B(a_2, x_2), \dots, B(a_n, x_n), B\left(a_{n+1}, \kappa - \sum_{j=1}^n x_j\right) \right\} \\ \text{s.t.} \quad & \sum_{j=1}^n x_j \leq \kappa \\ & x_j \geq 0, \quad j = 1(1)n. \end{aligned}$$

The set of feasible solutions of Problem 1, denoted by  $\mathbb{S}$  is a simplex of  $\mathbb{R}^n$ :

$$\mathbb{S} = \left\{ \mathbf{x} \in \mathbb{R}_+^n : \sum_{j=1}^n x_j \leq \kappa \right\}.$$

In addition, note that  $\mathbb{S}$  is a compact convex set of  $\mathbb{R}^n$ .

In the following,  $f_1$  and  $f_2$  designate the objective functions defined in Problem 1. The first objective of the bicriterion formulation is the minimization of the total lost traffic of the  $\mathcal{E}_B^{n+1}$  system, which is equivalent to the maximization of the total carried traffic. For this reason  $f_1$  will be designated as *efficiency objective function*. Accordingly, any global minimizer of  $f_1$  in the set of feasible solutions of Problem 1 is designated as *maximal efficiency solution*. The second objective of the bicriterion formulation is the minimization of a function defined as the blocking probability of the group with the worst grade of service. This objective is a criterion of grade of service fairness according to the MMF principle [2], and it will be seen that it is exactly an equity criterion since the minimum of  $f_2$  is achieved when all groups have equal blocking probabilities. For this reason,  $f_2$  will be designated as *equity objective function* and any global minimizer of  $f_2$  in the set of feasible solutions of Problem 1 will be designated as *maximal equity solution*.



### 3 Mathematical Properties

#### 3.1 The Blocking Vector Function

A fundamental task in order to solve Problem 1 is the establishment of relevant properties of the functions  $f_1$  and  $f_2$ , namely smoothness and convexity conditions. Indeed, the difficulties often encountered in the determination of those properties are well known in nonlinear programming practice and these properties play an important role in the resolution of the problem. Therefore due attention will be paid not only to the characterization of the objective functions, but also to the special structure of this problem in order to establish efficient numerical methods of resolution.

For  $j = 1(1)n + 1$ , let us introduce the blocking probability of each group of the  $\mathcal{E}_B^{n+1}$  system as a real function defined on  $\mathbb{S}$ :

$$\begin{aligned} B_j : \mathbb{S} \subset \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \mathbf{x} &\longmapsto B_j(\mathbf{x}), \end{aligned} \quad (11)$$

where,

$$\begin{aligned} B_j(\mathbf{x}) &= B(a_j, x_j), \quad j = 1(1)n \\ B_{n+1}(\mathbf{x}) &= B\left(a_{n+1}, \kappa - \sum_{j=1}^n x_j\right). \end{aligned} \quad (12)$$

Some basic properties of the  $B_j$  functions are now established.

##### Lemma 1

*$B_j$ ,  $j = 1(1)n + 1$ , are continuous functions on  $\mathbb{S}$  and twice continuously differentiable convex functions in  $\mathbb{S}^\circ$ .*

##### Proof:

Note that the Erlang-B function is continuous in variable  $x$  in the interval  $[0, +\infty[$  and a twice continuously differentiable function of  $x$  in  $\mathbb{R}_+$ . The non trivial part of this proof is the convexity of the functions  $B_j$  in  $\mathbb{S}^\circ$ , as a consequence of the known strict convexity of the Erlang-B function on variable  $x$  (see [4]). This is done by recognizing that the Hessians matrices  $\nabla^2 B_j$  are positive semidefinite.

For  $j = 1(1)n$  the gradient vector function  $\nabla B_j$  has only one non zero component in position  $j$  with value  $b'_j$ :

$$\nabla B_j = [0 \cdots 0 \ b'_j \ 0 \cdots 0]^T.$$

Consequently, the Hessian matrix  $\nabla^2 B_j$  has only one non zero entry at position  $(j, j)$  with value  $b''_j > 0$ . Therefore is a positive semidefinite matrix.

It remains to show that  $\nabla^2 B_{n+1}$  is a positive semidefinite matrix too. By differentiation, we have:

$$\nabla B_{n+1} = [-b'_{n+1} \ -b'_{n+1} \ -b'_{n+1} \ \cdots \ -b'_{n+1}]^T.$$

It is straightforward to show that  $\nabla^2 B_{n+1}$  is a matrix having all entries equal to  $b''_{n+1}$ . This matrix has only one non zero eigenvalue which is equal to  $nb''_{n+1} > 0$ . Consequently it is a positive semidefinite matrix.  $\square$

For each point  $\mathbf{x} \in \mathbb{S}$  the situation of statistical equilibrium of the system  $\mathcal{E}_B^{n+1}$  may be characterized through the blocking probabilities  $B_j$ ,  $j = 1(1)n+1$ . Those values may be seen as *state variables* of the system and the following definition is used in order to introduce the vector of such state variables:

$$\begin{aligned} \mathbf{B} : \mathbb{S} \subset \mathbb{R}^n &\longrightarrow \mathbb{R}^{n+1} \\ \mathbf{x} &\longmapsto \mathbf{B}(\mathbf{x}) = [B_1(\mathbf{x}), B_2(\mathbf{x}), \dots, B_n(\mathbf{x}), B_{n+1}(\mathbf{x})]^T. \end{aligned} \quad (13)$$

After defining that *blocking vector function*, we may have a more concise equivalent formulation of Problem 1:

$$\min_{\mathbf{x} \in \mathbb{S}} \left[ \begin{array}{c} \mathbf{a}^T \mathbf{B}(\mathbf{x}) \\ \|\mathbf{B}(\mathbf{x})\|_\infty \end{array} \right]. \quad (14)$$

### 3.2 The Efficiency Objective Function

#### Lemma 2

*The efficiency objective function has the following properties:*

- a)  $f_1$  is a continuous function on  $\mathbb{S}$  and a twice continuously differentiable function in  $\mathbb{S}^\circ$ ;
- b)  $f_1$  is a strictly convex function in  $\mathbb{S}$ ;
- c)  $f_1(\mathbf{x}) \in [\alpha B(\alpha, \kappa), \alpha[$ ,  $\forall \mathbf{x} \in \mathbb{S}$  where  $\alpha = \|\mathbf{a}\|_1 = \sum_{i=1}^{n+1} a_i$  (total offered traffic).

#### Proof:

Lemma 1 suffices to show a), since  $f_1$  is a linear combination of  $B_j$  functions. It remains to prove propositions b) and c):

- b) The gradient vector of  $f_1$  is

$$\nabla f_1(\mathbf{x}) = \begin{bmatrix} a_1 b'_1 - a_{n+1} b'_{n+1} \\ a_2 b'_2 - a_{n+1} b'_{n+1} \\ \vdots \\ a_n b'_n - a_{n+1} b'_{n+1} \end{bmatrix}. \quad (15)$$

The Hessian matrix of  $f_1$  is

$$\nabla^2 f_1(\mathbf{x}) = \begin{bmatrix} a_1 b''_1 + a_{n+1} b''_{n+1} & a_{n+1} b''_{n+1} & \cdots & a_{n+1} b''_{n+1} \\ a_{n+1} b''_{n+1} & a_2 b''_2 + a_{n+1} b''_{n+1} & \cdots & a_{n+1} b''_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1} b''_{n+1} & a_{n+1} b''_{n+1} & \cdots & a_n b''_n + a_{n+1} b''_{n+1} \end{bmatrix}. \quad (16)$$

The authors have shown in [4] that  $B''_x(a, x) > 0$  if  $x \geq 0$  for all positive values of  $a$ . Therefore, if  $\mathbf{x} \in \mathbb{S}$  we have:

$$\nabla^2 f_1(\mathbf{x}) = a_{n+1} b''_{n+1} \left[ E + \text{diag} \left[ \frac{a_1 b''_1}{a_{n+1} b''_{n+1}}, \frac{a_2 b''_2}{a_{n+1} b''_{n+1}}, \dots, \frac{a_n b''_n}{a_{n+1} b''_{n+1}} \right] \right]$$

where  $E \in \mathbb{R}^{n \times n}$  is a matrix having all its entries equal to 1. The essential observation is that  $b_j'' > 0$ ,  $j = 1(1)n + 1$  if  $\mathbf{x} \in \mathbb{S}$ . A little manipulation is needed for showing that  $\nabla^2 f_1$  is a positive definite matrix in this case. This may be done by proving that all its principal minors are positive — a classical result sometimes called Sylvester criterion. In fact, it may be shown by using induction on  $n$  and the Laplace theorem, that

$$\det(H_n) > 0$$

where  $H_n = E + \text{diag}(\mathbf{w})$  and  $\mathbf{w} \in \mathbb{R}_+^n$ .

- c) Since  $\kappa > 0$  it follows immediately that  $f_1(\mathbf{x}) < \alpha$ . The property  $f_1 \geq \alpha B(\alpha, \kappa)$  may be established as a direct consequence of the convexity of the Erlang-B function (see [26]).

□

Since  $f_1$  is a continuous function defined on the compact set  $\mathbb{S}$ ,  $f_1$  assumes its global minimum at a point  $\mathbf{x}^*$  in  $\mathbb{S}$ . The following lemma establishes additional results.

### Lemma 3

*$f_1$  has a unique local minimizer in  $\mathbb{S}$  which is also the unique global minimizer. Moreover, if  $\mathbf{x}^*$  is that point and  $\mathbf{x}^*$  belongs to the interior of  $\mathbb{S}$  then  $\nabla f_1(\mathbf{x}^*) = \mathbf{0}$ .*

#### Proof:

This lemma is a direct consequence of the strict convexity of function  $f_1$  established on Lemma 2. □  
A solution  $\mathbf{x} \in \mathbb{S}$  having some component near zero corresponds to a system  $\mathcal{E}_B^{n+1}$  which has some group with capacity allocated less than one circuit. In practice, such situations have no practical interest. Therefore, the interest in applications is restricted to the cases such that the maximal efficiency solution is attained in the interior of  $\mathbb{S}$ . Lemma 3 suggests the resolution of the stationarity system of equations for the numerical calculation of the solution. In the following, the maximal efficiency solution is denoted by  $\mathbf{x}^*$ .

## 3.3 Equity Objective Function

### Lemma 4

*The equity objective function has the following properties:*

- a)  $f_2$  is a continuous function in  $\mathbb{S}$ .
- b)  $f_2$  is a convex function in  $\mathbb{S}^\circ$ ;

#### Proof:

- a) Note that  $f_2(\mathbf{x}) = \|\mathbf{B}(\mathbf{x})\|_\infty$  and a norm is a continuous mapping. Additionally, the vector function  $\mathbf{B}$  is a continuous function in  $\mathbb{S}$ ;  $f_2$  is defined as a composition of two continuous functions.

- b) Note that  $f_2 = \max\{B_1, \dots, B_n, B_{n+1}\}$  and  $B_j$  are convex functions on  $\mathbb{S}$ . Consequently  $f_2$  is a convex function (see for example [22, pag.78]).

□

Since  $f_2$  is a continuous function defined on a bounded and closed set  $\mathbb{S}$ , the Weierstrass theorem may be used to show that  $f_2$  assumes its global minimum at a point  $\mathbf{x}^{**}$  in  $\mathbb{S}$ . Note that  $f_2$  is not a differentiable function in  $\mathbb{S}$ . Minimization of non differentiable functions is an hard task even for the case of convex functions. The following lemma establishes an important result, showing that the (unique) maximal equity solution may be calculated by solving a system of smooth nonlinear equations.

### Lemma 5

$f_2$  takes on its unique global minimum over  $\mathbb{S}$ , at a point  $\mathbf{x}^{**}$ , such that:

- a)  $\mathbf{x}^{**}$  is the unique solution in  $\mathbb{S}$  of the following system of equations:

$$B(a_1, x_1^{**}) = B(a_2, x_2^{**}) = \dots = B(a_n, x_n^{**}) = B\left(a_{n+1}, \kappa - \sum_{j=1}^n x_j^{**}\right); \quad (17)$$

- b)  $\mathbf{x}^{**}$  is an interior point of  $\mathbb{S}$ , that is:

$$\begin{aligned} \sum_{j=1}^n x_j^{**} &< \kappa, \\ x_j^{**} &> 0, \quad j = 1(1)n. \end{aligned} \quad (18)$$

### Proof:

In order to avoid unwanted formal complications, let us introduce the following single criterion problem:

$$\begin{aligned} \min_{\mathbf{y} \in \mathbb{R}^{n+1}} \quad & g(\mathbf{y}) = \max\{B(a_1, y_1), B(a_2, y_2), \dots, B(a_n, y_n), B(a_{n+1}, y_{n+1})\} \\ \text{s.t.} \quad & \sum_{j=1}^{n+1} y_j = \kappa \\ & y_j \geq 0, \quad j = 1(1)n+1. \end{aligned} \quad (19)$$

The set of feasible solutions of this problem will be designated as  $\mathcal{Y} \subset \mathbb{R}_{\oplus}^{n+1}$ . Using an obvious change of variable it is straightforward to verify that we only need to prove that the optimal solution of (19) (denoted by  $\mathbf{y}^{**}$ ) is unique and

$$B(a_1, y_1^{**}) = B(a_2, y_2^{**}) = \dots = B(a_n, y_n^{**}) = B(a_{n+1}, y_{n+1}^{**}), \quad (20)$$

$$y_j^{**} > 0, \quad j = 1(1)n+1. \quad (21)$$

Introducing the notation  $\mathcal{I} = \{1, 2, \dots, n, n+1\}$ , let us now define the following sets associated with a generic point  $\mathbf{y} \in \mathcal{Y}$ :

$$\begin{aligned} \mathcal{L}(\mathbf{y}) &= \{j \in \mathcal{I} : B(a_j, y_j) < g(\mathbf{y})\}, \\ \mathcal{M}(\mathbf{y}) &= \{j \in \mathcal{I} : B(a_j, y_j) = g(\mathbf{y})\}. \end{aligned}$$

It is easy to see that for all  $\mathbf{y}$  in  $\mathcal{Y}$ ,  $\mathcal{M}(\mathbf{y})$  is not an empty set. Furthermore,  $\mathcal{L}(\mathbf{y}) \cap \mathcal{M}(\mathbf{y}) = \emptyset$  and  $\mathcal{L}(\mathbf{y}) \cup \mathcal{M}(\mathbf{y}) = \mathcal{I}$ . Additionally, note that if  $j \in \mathcal{L}(\mathbf{y})$  then  $y_j > 0$ , that is:

$$\forall j \in \mathcal{L}(\mathbf{y}), \quad y_j > 0. \quad (22)$$

Proposition (22) may be easily proved by *reductio ad absurdum*. Indeed, if  $y_j = 0$  then  $B(a_j, y_j) = 1 \geq g(\mathbf{y})$ .

The proof will be divided in two parts:

- (i) Firstly, we establish that any global minimizer of  $g$  in  $\mathcal{Y}$  is a solution of the system of equations (20). We shall use *reductio ad absurdum*. Suppose the contrary, that is let  $\hat{\mathbf{y}}$  designate a global minimizer in  $\mathcal{Y}$  which does not satisfy (20). Then,  $\mathcal{L}(\hat{\mathbf{y}})$  is not an empty set. It is then possible to take an integer value  $i \in \mathcal{L}(\hat{\mathbf{y}})$  and a real value  $\epsilon > 0$  in order to define a point  $\check{\mathbf{y}} \in \mathbb{R}^{n+1}$ , having the following components:

$$\begin{aligned} \check{y}_i &= \hat{y}_i - \epsilon, \\ \check{y}_j &= \hat{y}_j, \quad \forall j \in \mathcal{L}(\hat{\mathbf{y}}) \setminus \{i\}, \\ \check{y}_j &= \hat{y}_j + \frac{\epsilon}{|\mathcal{M}(\hat{\mathbf{y}})|}, \quad \forall j \in \mathcal{M}(\hat{\mathbf{y}}), \end{aligned}$$

where  $|\mathcal{M}(\hat{\mathbf{y}})|$  denotes the size of the set  $\mathcal{M}(\hat{\mathbf{y}})$ . Taking into account (22), if  $\epsilon < \hat{y}_i$  then  $\check{\mathbf{y}} \in \mathcal{Y}$  since we have  $\sum_{j=1}^{n+1} \check{y}_j = \kappa$  and  $\check{y}_j \geq 0$ ,  $j = 1(1)n+1$ . Furthermore a transition from point  $\hat{\mathbf{y}}$  to point  $\check{\mathbf{y}}$  implies that only group  $i$  increases its blocking. Additionally, blocking decreases in all groups with indices in  $\mathcal{M}(\hat{\mathbf{y}})$ . By the continuity of the Erlang-B function it is possible to choose a real value  $\epsilon$  ( $0 < \epsilon < \hat{y}_i$ ), sufficiently small so that the blocking in group  $i$  remains non maximal at point  $\check{\mathbf{y}}$ . It is then obvious:

$$g(\check{\mathbf{y}}) < g(\hat{\mathbf{y}}),$$

implying that  $\hat{\mathbf{y}}$  is not a global minimizer of the  $g$  function on  $\mathcal{Y}$ , which contradicts the initial assumption. Therefore, if  $\mathbf{y}^{**}$  is a global minimizer then  $\mathcal{L}(\mathbf{y}^{**})$  must be an empty set. That fact implies that  $\mathcal{M}(\mathbf{y}^{**}) = \mathcal{I}$  which leads to the conclusion that  $\mathbf{y}^{**}$  satisfies equations (20).

Note that this system has at least one solution in  $\mathcal{Y}$  — if we suppose the contrary then  $g(\mathbf{y})$  does not have its minimum on  $\mathcal{Y}$  which is absurd.

- (ii) Finally, we have to show that equations (20) have a unique solution  $\mathbf{y}^{**}$  in  $\mathcal{Y}$  satisfying (21). Once more, we shall use *reductio ad absurdum*. Let us suppose that two distinct points  $\mathbf{y}$  and  $\mathbf{y}'$  both in  $\mathcal{Y}$ , satisfy equations (20). Let  $\beta = B_j(a_j, y_j)$ ,  $j = 1(1)n+1$  and  $\beta' = B_j(a_j, y'_j)$ ,  $j = 1(1)n+1$ . Since the Erlang-B function is a monotone function in  $x$ , it is obvious that  $\beta \neq \beta'$ . Suppose for example that  $\beta < \beta'$ . This fact implies that  $y_j > y'_j$ ,  $j = 1(1)n+1$  which is an absurd proposition. Indeed, since  $\mathbf{y}$  and  $\mathbf{y}'$  are points in  $\mathcal{Y}$  we have  $\|\mathbf{y}\|_1 = \|\mathbf{y}'\|_1 = \kappa$ . The conclusion is that the hypothesis of distinct solutions of equations (20) on  $\mathcal{Y}$  conducts to a contradiction, therefore the uniqueness of the solution is proved.

It remains to show that the solution  $\mathbf{y}^{**}$  of (20) in  $\mathcal{Y}$  does satisfy (21). Once again, assuming the contrary we have a contradiction. Let us admit that vector  $\mathbf{y}^{**}$  has some zero component. It follows that the corresponding blocking is equal to one and due to equations (20) all groups must have blocking equal to one. This situation is impossible, since  $\kappa > 0$ .

□

As discussed above function  $f_2$  gives an adequate measure of the grade of service of the  $\mathcal{E}_B^{n+1}$  system. Lemma 5 proves that the function  $f_2$  is an equity criterion too, since its minimum is achieved in a situation such that there is a completely uniformity of the grade of service of all groups of the  $\mathcal{E}_B^{n+1}$  system.

### 3.4 The Maximal Efficiency Solution

The maximal efficiency solution may be an interior point or a boundary point of  $\mathbb{S}$ . As discussed above the cases in which the maximal efficiency solution is attained on  $\mathbb{S}^\circ$ , have special interest in the applications. Lemma 3 establishes uniqueness of the maximal efficiency solution in this case and suggests the resolution of the stationarity system of equations for the numerical calculation of the solution. In this section an algorithmic approach is proposed for the typical case  $\mathbf{x}^* \in \mathbb{S}^\circ$ . At the end of this section we shall give some indications about the numerical calculation of the maximal efficiency solutions when  $\mathbf{x}^* \in \partial\mathbb{S}$ , where  $\partial\mathbb{S}$  denotes the boundary point of  $\mathbb{S}$ .

#### 3.4.1 Interior Point Maximal Efficiency Solution

Denoting by  $\Phi(\mathbf{x})$  the gradient vector of  $f_1$  defined on the interior points of  $\mathbb{S}$ , we have:

$$\begin{aligned} \Phi : \mathbb{S}^\circ \subset \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto \Phi(\mathbf{x}) = [\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \dots, \phi_n(\mathbf{x})]^T, \end{aligned}$$

where

$$\phi_i(\mathbf{x}) = a_1 B'_x(a_i, x_i) - a_{n+1} B'_x\left(a_{n+1}, \kappa - \sum_{j=1}^n x_j\right) = a_i b'_i - a_{n+1} b'_{n+1}, \quad i = 1(1)n \quad (23)$$

Our problem reduces to solve  $\Phi(\mathbf{x}) = \mathbf{0}$  on  $\mathbb{S}$  using a numerical method. A Newton-Raphson method is proposed. The application of Newton method is discussed on Appendix A. Given an iterate  $\mathbf{x}$ , Newton's method generates the next iterate  $\mathbf{x}^+ = \mathbf{x} + \mathbf{y}$  by solving the linear system  $\Phi'(\mathbf{x})\mathbf{y} = -\Phi(\mathbf{x})$ , where  $\Phi'(\mathbf{x})$  is the Jacobian matrix of  $\Phi$  evaluated at  $\mathbf{x}$ . Some tedious manipulations yield  $\Phi'$ :

$$\begin{bmatrix} a_1 b''_1 + a_{n+1} b''_{n+1} & a_{n+1} b''_{n+1} & a_{n+1} b''_{n+1} & \cdots & a_{n+1} b''_{n+1} \\ a_{n+1} b''_{n+1} & a_2 b''_2 + a_{n+1} b''_{n+1} & a_{n+1} b''_{n+1} & \cdots & a_{n+1} b''_{n+1} \\ a_{n+1} b''_{n+1} & a_{n+1} b''_{n+1} & a_3 b''_3 + a_{n+1} b''_{n+1} & \cdots & a_{n+1} b''_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n+1} b''_{n+1} & a_{n+1} b''_{n+1} & a_{n+1} b''_{n+1} & \cdots & a_n b''_n + a_{n+1} b''_{n+1} \end{bmatrix}.$$

For solving the linear system, we also need to have the general form of the vector  $\Phi(\mathbf{x})$ . Let us remember that  $B''_x(a, x)$  is positive if  $x \geq 0$  for all  $a \in \mathbb{R}_+$ . Providing that  $b''_{n+1} \neq 0$  the linear

system  $\Phi'(\mathbf{x})\mathbf{y} = -\Phi(\mathbf{x})$  is equivalent to

$$[E + \text{diag}(w_\Phi)] \mathbf{y} = b_\Phi, \quad (24)$$

where:

$$w_\Phi = \left[ \frac{a_1 b_1''}{a_{n+1} b_{n+1}''}, \frac{a_2 b_2''}{a_{n+1} b_{n+1}''}, \frac{a_3 b_3''}{a_{n+1} b_{n+1}''}, \dots, \frac{a_n b_n''}{a_{n+1} b_{n+1}''} \right]^T, \quad (25)$$

$$b_\Phi = \left[ \frac{a_{n+1} b_{n+1}' - a_1 b_1'}{a_{n+1} b_{n+1}'}, \frac{a_{n+1} b_{n+1}' - a_2 b_2'}{a_{n+1} b_{n+1}'}, \dots, \frac{a_{n+1} b_{n+1}' - a_n b_n'}{a_{n+1} b_{n+1}'} \right]^T. \quad (26)$$

Since  $w_\Phi \in \mathbb{R}_+^n$ , an efficient algorithm presented in Appendix A solves the linear system and generates a Newton sequence for solving the equation  $\Phi(\mathbf{x}) = \mathbf{0}$ . In order to calculate the components of the vectors  $w_\Phi$  and  $b_\Phi$  in each iteration of the Newton Method, the first two derivatives of the Erlang-B function in the variable  $x$  are calculated by using an efficient method proposed by the authors [12, 13].

It remains to define the initial approximation for starting the Newton sequence. The point which is the geometric center of the simplex  $\mathbb{S}$ , is considered for this purpose:

$$x_j^{(0)} = \frac{\kappa}{\sum_{i=1}^{n+1} a_i} a_j, \quad j = 1(1)n. \quad (27)$$

Note that this point is the exact solution if all the offered traffics are equal. This is easily proved by symmetry arguments.

### 3.4.2 Boundary Point Maximal Efficiency Solution

If  $\mathbf{x}^* \in \mathbb{S}$  is the maximal efficiency solution of Problem 1 then the associated vector of group capacities on  $\mathcal{E}_B^{n+1}$  will be denoted by  $\bar{\mathbf{x}}^*$ , with components:

$$\begin{aligned} x_j^* & \quad \text{for } j = 1(1)n, \\ x_{n+1}^* & = \kappa - \sum_{j=1}^n x_j^*. \end{aligned}$$

If  $\mathbf{x}^* \in \partial\mathbb{S}$ , then

$$\exists j \in \{1, 2, \dots, n, n+1\} : x_j^* = 0,$$

which means that the corresponding  $\mathcal{E}_B^{n+1}$  system has some group(s) with zero allocated capacity. The following lemma is very important for the characterization of maximal efficiency solutions on boundary points.

#### Lemma 6

*If  $\mathbf{x}^* \in \partial\mathbb{S}$  is a global minimizer of  $f_1$  on  $\mathbb{S}$ , then the associated vector of group capacities  $\bar{\mathbf{x}}^*$  on  $\mathcal{E}_B^{n+1}$  has at least one zero component  $x_k^*$  for  $k \in \{1, 2, \dots, n, n+1\}$ . Furthermore, if  $k < n+1$ , then*

$$x_k^* = 0 \implies x_{k+1}^* = 0.$$

**Proof:**

Assuming  $k < n + 1$  we only have to prove that if  $x_k^*$  is zero, then  $x_{k+1}^*$  must be zero. If  $a_k = a_{k+1}$ , an obvious symmetry argument suffices to show that  $x_k^* = x_{k+1}^*$ .

Due to assumption (10) it remains to analyze a case such that  $a_k > a_{k+1}$ . First, note that

$$f_1(\mathbf{x}^*) = \sum_{j=1}^{n+1} a_j B(a_j, x_j^*),$$

and,

$$x_k^* = 0 \implies a_k B(a_k, x_k^*) = a_k.$$

Let us assume that  $x_k^*$  is zero, and  $x_{k+1}^* = \epsilon > 0$ . That is, group  $k$  has zero allocated capacity and group  $k + 1$  has  $\epsilon > 0$  allocated capacity. Since  $a_k > a_{k+1}$ , then it is possible to decrease the objective function by transferring the capacity  $\epsilon$  of group  $k + 1$  to group  $k$  which is an absurd because  $\mathbf{x}^*$  is the maximal efficiency solution. To prove that  $f_1$  decreases in that situation, one only needs to take into account the following basic property of an Erlang-B group: the carried traffic function  $A_c(a, x)$  defined by (7) is a strictly increasing function in  $a$  (see [10, Lemma 3.14, pp.18]).  $\square$

Perhaps the most popular approach to solving linear constrained convex programming problems is to use a so-called *active-set strategy*, which is based on the following idea. If a feasible point and the set of active constraints on the optimal solution were known, the solution could be computed directly as described in the Sub-sub-section 3.4.1. As in linear programming the hard part is to identify the set of active constraints on the optimal solution. Since these are unknown, a prediction of the active set — called the working set — is developed which is used to compute the search direction, and then the working set is changed as the iterations proceed.

Lemma 6 is very important in order to develop a Newton algorithm that is combined with an active constraints strategy of the classical type used for nonnegativity constraints. The basis of the proposed algorithm is now described. Start the process as described in Sub-section 3.4.1, that is solve  $\nabla f_1(\mathbf{x}) = \mathbf{0}$  by the Newton method. If  $\mathbf{x}^*$  is not an interior point of  $\mathbb{S}$ , then the Newton sequence falls out the feasible region. When this situation is detected we may conclude, by using Lemma 6, that  $x_{n+1}^* = 0$ . We have now a reduced maximal efficiency problem of allocation of a total of  $\kappa$  circuits to  $n$  groups instead of  $n + 1$  groups, as initially. The process may be repeated in the same manner until a Newton sequence for a reduced problem converges to an interior point of the reduced feasible set.

## 4 The Maximal Equity Solution

The proposed approach to the numerical calculation of the maximal equity solution is similar to the one presented in Sub-section 3.4.1 for the calculation of the maximal efficiency solution in the interior of  $\mathbb{S}$ .

Let us introduce the vector function:

$$\begin{aligned} \Psi : \mathbb{S} \subset \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto \Psi(\mathbf{x}) = [\psi_1(\mathbf{x}), \psi_2(\mathbf{x}), \dots, \psi_n(\mathbf{x})]^T, \end{aligned}$$



where

$$\psi_i(\mathbf{x}) = B(a_i, x_i) - B\left(a_{n+1}, \kappa - \sum_{j=1}^n x_j\right) = b_i - b_{n+1}, \quad i = 1(1)n. \quad (28)$$

Lemma 5 establishes that the maximal equity solution may be computed by solving the vector equation  $\Psi(\mathbf{x}) = \mathbf{0}$ . Since  $\Psi$  is a twice continuously differential function in the interior of  $\mathbb{S}$  a Newton method is proposed. If  $\Psi'$  denotes the Jacobian matrix of  $\Psi$ , in each iteration we need to solve the linear system  $\Psi'(\mathbf{x}) \mathbf{y} = -\Psi(\mathbf{x})$ , which is equivalent to:

$$\begin{bmatrix} b'_1 + b'_{n+1} & b'_{n+1} & b'_{n+1} & \cdots & b'_{n+1} \\ b'_{n+1} & b'_2 + b'_{n+1} & b'_{n+1} & \cdots & b'_{n+1} \\ b'_{n+1} & b'_{n+1} & b'_3 + b'_{n+1} & \cdots & b'_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b'_{n+1} & b'_{n+1} & b'_{n+1} & \cdots & b'_n + b'_{n+1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_{n+1} - b_1 \\ b_{n+1} - b_2 \\ b_{n+1} - b_3 \\ \vdots \\ b_{n+1} - b_n \end{bmatrix}$$

Let us remember that  $B'_x(a, x) < 0$  for all  $a, x \in \mathbb{R}_+$ . Since  $b'_{n+1} \neq 0$ , the linear system is equivalent to

$$[E + \text{diag}(w_\Psi)] \mathbf{y} = b_\Psi, \quad (29)$$

where:

$$w_\Psi = \left[ \frac{b'_1}{b'_{n+1}}, \frac{b'_2}{b'_{n+1}}, \frac{b'_3}{b'_{n+1}}, \dots, \frac{b'_n}{b'_{n+1}} \right]^T, \quad (30)$$

$$b_\Psi = \left[ \frac{b_{n+1} - b_1}{b'_{n+1}}, \frac{b_{n+1} - b_2}{b'_{n+1}}, \frac{b_{n+1} - b_3}{b'_{n+1}}, \dots, \frac{b_{n+1} - b_n}{b'_{n+1}} \right]^T. \quad (31)$$

Since  $w_\Psi \in \mathbb{R}_+^n$  the efficient algorithm presented in Appendix A solves the linear system and generates a Newton sequence for solving the equation  $\Psi(\mathbf{x}) = 0$ .

As for the case of the computation of the maximal efficiency solution, the initial approximation proposed is the geometric center of simplex  $\mathbb{S}$  defined by expression (27). Symmetry arguments may be used to show that this point is the maximal equity solution if all the offered traffics are equal.

## 5 The Conflict between Efficiency and Equity

As discussed before, the maximal efficiency solution and the maximal equity solution are the same if  $a_j = a$ ,  $j = 1(1)n + 1$ . In this case Problem 1 is trivial. In other cases, we have two conflicting objectives in Problem 1. The proof of this proposition is the main task of this section.

First, note that there is an obvious conflict if the maximal efficiency solution is a boundary point of  $\mathbb{S}$ , since the maximal equity solution is an interior point of  $\mathbb{S}$ .

It remains to analyze the case for which the maximal efficiency solution is an interior point of  $\mathbb{S}$ . In [10, Theorem 6] the resource allocation between only two Erlang-B groups (that is

an  $\mathcal{E}_B^2$  system is analyzed). In this case we have only one decision variable, and Problem 1 is formulated as

$$\begin{aligned} \min_{x_1} \quad & f_1(x_1) = a_1 B(a_1, x_1) + a_2 B(a_2, \kappa - x_1) \\ \min_{x_1} \quad & f_2(x_1) = \max \{B(a_1, x_1), B(a_2, \kappa - x_1)\} \\ \text{s.t.} \quad & x_1 \in [0, \kappa] \end{aligned} \quad (32)$$

Denoting the maximal efficiency solution by  $x_1^* \in [0, \kappa]$  and the maximum equity solution by  $x_1^{**} \in ]0, \kappa[$ , the statement of Theorem 6 of [10] is the proposition:

$$a_1 > a_2 \implies \frac{df_1}{dx_1}(x_1^{**}) < 0, \quad \forall \kappa \in \mathbb{R}_+. \quad (33)$$

Proposition (33) has several important implications on the analysis of  $\mathcal{E}_B^{n+1}$  system.

**Lemma 7**

*If  $a_1 > a_2$ , then  $B(a_1, x_1^*) < B(a_2, \kappa - x_1^*)$ ,  $\forall \kappa \in \mathbb{R}_+$ .*

**Proof:**

Since  $f_1$  is a smooth convex function in the interval  $[0, k]$  the derivative  $f_1'$  does not decrease. By using proposition (33) we have  $f_1'(x_1^{**}) < 0$ , therefore

$$f_1'(x_1) < 0, \quad \forall x_1 \in [0, x_1^{**}].$$

Thence,  $f_1$  assumes its minimum value on  $[0, k]$  in some point in the interval  $]x_1^{**}, \kappa]$ , that is

$$x_1^* > x_1^{**}.$$

In view of Lemma 5 the result follows from

$$\begin{aligned} B(a_1, x_1^{**}) &= B(a_2, \kappa - x_1^{**}), \\ B(a_1, x_1) &< B(a_2, \kappa - x_1), \quad \forall x_1 > x_1^{**}. \end{aligned}$$

□

Lemma 7 shows the conflict between efficiency and equity for the case of a  $\mathcal{E}_B^2$  system. The generalization of this result for a system  $\mathcal{E}_B^{n+1}$ ,  $n > 1$  needs a preparatory result, which may be seen as an application of the classical *Bellman optimality principle* to the  $\mathcal{E}_B^{n+1}$  system.

**Lemma 8**

*Any set of  $l < n+1$  groups of a maximal efficiency system  $\mathcal{E}_B^{n+1}$  is a maximal efficiency system  $\mathcal{E}_B^l$ , providing the total capacity allocated to system  $\mathcal{E}_B^l$  is a positive value.*

**Proof:**

If the capacity allocated to the system  $\mathcal{E}_B^l$  is a positive value, then the problem of optimal efficiency resource allocation has a solution. The proof is easily made by using *reductio ad absurdum*. Indeed, suppose that the total carried traffic in  $\mathcal{E}_B^l$  system is not maximal. This means that it is possible to reallocate capacity among its  $l$  Erlang-B groups, increasing the total carried traffic in  $\mathcal{E}_B^l$  system. Consequently the total carried traffic in  $\mathcal{E}_B^{n+1}$  is increased too, which is an absurd situation. □

We are now in a position to prove that the two objectives of Problem 1 are conflicting in the general case.

**Lemma 9**

If  $\mathbf{x}^*$  denotes the maximal efficiency solution of problem 1 and  $b_j^*$ ,  $j = 1(1)n + 1$  the corresponding blocking probabilities on the server groups, then

a) If  $a_1 > a_2 > \dots > a_n > a_{n+1}$ , and  $\mathbf{x}^* \in \mathbb{S}^\circ$ , then:

$$b_1^* < b_2^* < \dots < b_n^* < b_{n+1}^* ;$$

b) For the general case, that is  $a_1 \geq a_2 \geq \dots \geq a_n \geq a_{n+1}$ , and  $\mathbf{x}^* \in \mathbb{S}$ , then:

$$b_j^* \leq b_{j+1}^*, \quad j = 1(1)n,$$

and equality  $b_m^* = b_{m+1}^*$  only holds for some  $m \in \{1, 2, \dots, n\}$ , if and only if:

$$(a_m = a_{m+1}) \vee (\bar{x}_m^* = 0).$$

**Proof:**

- a) Proposition a) is a direct consequence of Lemmas 7 and 8. Actually, if  $\mathbf{x}^* \in \mathbb{S}^\circ$  then all groups in the system  $\mathcal{E}_B^{n+1}$  have positive allocated capacity. Thence, by using Lemma 8 any subsystem  $\mathcal{E}_B^2$  consisting of a pair of groups of the original system  $\mathcal{E}_B^{n+1}$ , is characterized by having maximal efficiency. Applying that conclusion to the first two groups of the system  $\mathcal{E}_B^{n+1}$  and by using Lemma 8 it is concluded that  $b_1^* < b_2^*$ . An obvious induction completes the proof.
- b) On the other hand, by symmetry arguments, if  $a_m = a_{m+1}$ , then  $b_m^* = b_{m+1}^*$ . Moreover,  $x_m^* = 0$  implies  $x_{m+1}^* = 0$  by using Lemma 6, that is  $b_m^* = b_{m+1}^* = 1$ . In all the remaining cases we may use the argument used in a) to conclude that  $b_m^* < b_{m+1}^*$ .

□

Lemma 9 characterizes the conflicting nature of the two objectives of problem 1. Furthermore, we may conclude that the situation of maximal efficiency in system  $\mathcal{E}_B^{n+1}$  leads to a situation of better grade of service in the groups having greater offered traffic.

## 6 Resolution Approach

### 6.1 Characterization of the Pareto Optimal Solutions

Unlike traditional mathematical programming with a single objective function, in typical multiobjective optimization problems, an optimal solution, in the sense that it minimizes all the objective functions simultaneously, does not exist. Thence we are dealing with conflicts among objectives in decision making problems with multiple objectives and we are seeking for Pareto optimal solutions, that is, solutions such that it is not possible to improve one objective function without worsening at least one of the other objective functions. It may be shown that

such conflict exists between the value of the two objectives of Problem 1, except for the trivial case  $a_j = a$ ,  $j = 1(1)n + 1$  (symmetry of  $\mathcal{E}_B^{n+1}$  system). Applying classical results of bicriterion convex programming (see for example [1] and [25]), Pareto optimal solutions of Problem 1 are characterized in this section as solutions of a parametric single criterion convex problem with linear constraints. Additionally it is shown that the approach proposed in Sub-sub-section 3.4.2 for computing maximal efficiency solutions on boundary points may be applied for computing Pareto Optimal solutions of Problem 1.

Firstly, note that from the uniqueness of the maximal equity solution  $\mathbf{x}^{**}$  it is obvious that  $\mathbf{x}^{**}$  is a Pareto optimal solution of Problem 1. In the same manner, the maximal efficiency solution  $\mathbf{x}^*$  is unique and therefore is Pareto optimal too.

Let us now introduce the notation

$$\begin{aligned}\beta^* &= f_2(\mathbf{x}^*), \\ \beta^{**} &= f_2(\mathbf{x}^{**}).\end{aligned}$$

Using [25, Theorem 2],  $\hat{\mathbf{x}} \in \mathbb{S}$  is a Pareto Optimal solution of Problem 1 if and only if  $\hat{\mathbf{x}} \in \mathbb{S}$  solves the following parametric monocriterion convex program.

**Problem 2 (First Single Criterion Parametric Formulation of Problem 1)**

Given  $n \in \mathbb{N}$ ,  $\kappa \in \mathbb{R}_+$  and a traffic offered vector  $\mathbf{a} \in \mathbb{R}_+^{n+1}$ , solve the following program for  $\beta \in [\beta^{**}, \beta^*]$ :

$$\begin{aligned}\min_{\mathbf{x}} \quad & f_1(\mathbf{x}) = \sum_{j=1}^n a_j B(a_j, x_j) + a_{n+1} B\left(a_{n+1}, \kappa - \sum_{j=1}^n x_j\right) \\ \text{s.t.} \quad & f_2(\mathbf{x}) \leq \beta \\ & \sum_{j=1}^n x_j \leq \kappa \\ & x_j \geq 0, \quad j = 1(1)n\end{aligned}$$

Therefore, the set  $\mathcal{N}$  of Pareto optimal solutions of problem 1 may be defined as

$$\mathcal{N} = \left\{ \hat{\mathbf{x}} \in \mathbb{S} : \mathbf{a}^T \mathbf{B}(\hat{\mathbf{x}}) = \min_{\mathbf{x} \in \mathbb{S}_\beta} \mathbf{a}^T \mathbf{B}(\mathbf{x}), \beta \in [\beta^{**}, \beta^*] \right\}.$$

Denoting by  $\mathbf{x}_\beta^*$  the solution of Problem 2 for some value of  $\beta \in [\beta^{**}, \beta^*]$ , it is easy to see that:

- (i)  $\mathbf{x}_\beta^* = \mathbf{x}^{**}$  if  $\beta = \beta^{**}$ ;
- (ii)  $\mathbf{x}_\beta^* = \mathbf{x}^*$  if  $\beta \geq \beta^*$ ;
- (iii) The set of points  $\mathbf{x}_\beta^* \in \mathbb{S}$  for  $\beta \in [\beta^{**}, \beta^*]$  is a line in  $\mathbb{R}^n$  having extreme points  $\mathbf{x}^{**}$  and  $\mathbf{x}^*$ .

Let us now examine the constraint  $f_2(\mathbf{x}) \leq \beta$  of Problem 2. It may be written:

$$f_2(\mathbf{x}) \leq \beta \iff \left[ (B(a_i, x_i) \leq \beta, \ i = 1(1)n) \bigwedge (B(a_{n+1}, \kappa - \sum_{j=1}^n x_j) \leq \beta) \right].$$

In view of the monotonicity of the Erlang-B function in  $x$ , unique values  $x_1^\beta, x_2^\beta, \dots, x_{n+1}^\beta$ , exist such that:

$$\left( B(a_i, x_i^\beta) = \beta, \ i = 1(1)n \right) \bigwedge B(a_{n+1}, x_{n+1}^\beta) = \beta \quad (34)$$

Computation of such values is a classical numerical problem related to the Erlang-B function. Several methods are proposed in the literature for this purpose (see for example [19] and [21]).

After calculating such values, it is straightforward:

$$f_2(\mathbf{x}) \leq \beta \iff \left[ (x_i \geq x_i^\beta, \ i = 1(1)n) \bigwedge \left( \sum_{j=1}^n x_j \leq \kappa - x_{n+1}^\beta \right) \right] \iff x_i \geq x_i^\beta, \ i = 1(1)n+1 \quad (35)$$

Since  $x_j^\beta > 0$ ,  $j = 1(1)n + 1$  the equivalence (35) gives the following formulation of Problem 2.

### Problem 3 (Second Parametric Single Criterion Formulation of Problem 1)

Given  $n \in \mathbb{N}$ ,  $\kappa \in \mathbb{R}_+$  and a traffic offered vector  $\mathbf{a} \in \mathbb{R}_+^{n+1}$ , solve the following program for  $\beta \in [\beta^{**}, \beta^*]$ , where the  $x_j^\beta$ ,  $j = 1(1)n + 1$  are obtained from (34):

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_1(\mathbf{x}) = \sum_{j=1}^n a_j B(a_j, x_j) + a_{n+1} B(a_{n+1}, \kappa - \sum_{j=1}^n x_j) \\ \text{s.t.} \quad & x_j \geq x_j^\beta, \ j = 1(1)n \\ & \sum_{j=1}^n x_j \leq \kappa - x_{n+1}^\beta \end{aligned}$$

Problem 3 is a parametric single criterion convex problem with linear constraints. Denoting by  $\mathbb{S}_\beta$  the feasible region for each value of the parameter  $\beta$ , note that  $\mathbb{S}_\beta = \{\mathbf{x}^{**}\}$  for  $\beta = \beta^{**}$ . If  $\beta \in ]\beta^{**}, \beta^*]$  then  $\mathbb{S}_\beta$  is a simplex of  $\mathbb{R}^n$ . Furthermore, if  $\beta$  and  $\beta'$  are values in  $[\beta^{**}, \beta^*]$  such that  $\beta' < \beta$ , then:

$$\mathbb{S}_{\beta'} \subset \mathbb{S}_\beta.$$

## 6.2 Algorithm for Traveling on $\mathcal{N}$

As shown in [25, Theorem 1],  $f_2(\mathbf{x}) \leq \beta$  is an active constraint at the optimal solution of Problem 2. That is, if  $\mathbf{x}_\beta^*$  solves Problem 2, then  $f_2(\mathbf{x}_\beta^*) = \beta$ . By expression (35) it follows that,

$$\exists j \in \{1, 2, \dots, n, n+1\} : x_j^* = x_j^\beta. \quad (36)$$

Let us establish a convention for labeling constraints of Problem 3. Constraint  $x_j \geq x_j^\beta$  is labeled with number  $j$  for  $j = 1(1)n$ . Constraint  $\sum_{j=1}^n x_j \leq \kappa - x_{n+1}^\beta$  is labeled with number  $n+1$ .  $\mathcal{A}(\beta)$  denotes the set of numbers which are labels of the active constraints of Problem 3 at the optimal solution, for a generic value of the parameter  $\beta \in [\beta^{**}, \beta^*]$ .

**Lemma 10**

If  $m < n + 1$ , and  $\mathcal{E}_B^{n+1}$  is a maximal efficiency system then

$$m \in \mathcal{A}(\beta) \implies m + 1 \in \mathcal{A}(\beta).$$

**Proof:**

From [25, Theorem 1] this means that the system  $\mathcal{E}_B^2$  composed of groups  $m$  and  $m + 1$  is not in the situation of maximal efficiency (since efficiency improvement might be obtained by transferring capacity from group  $m$  to group  $m + 1$ ), which is absurd by Lemma 8 since  $\mathcal{E}_B^{n+1}$  is in a situation of maximal efficiency.  $\square$

By using (36), it may be written:

$$\mathcal{A}(\beta) \neq \emptyset, \quad \forall \beta \in [\beta^{**}, \beta^*]. \quad (37)$$

This proposition and Lemma 10 are sufficient to conclude that

$$n + 1 \in \mathcal{A}(\beta), \quad \forall \beta \in [\beta^{**}, \beta^*]. \quad (38)$$

Furthermore, it is easy to see that

$$\begin{aligned} \mathcal{A}(\beta^{**}) &= \{1, 2, \dots, n, n + 1\}, \\ \mathcal{A}(\beta^*) &= \{n + 1\} \text{ if } (a_{n+1} < a_n) \wedge (x_n^* > 0). \end{aligned}$$

For calculating a Pareto optimal solution corresponding to a certain  $\beta \in ]\beta^{**}, \beta^*[$  the following calculation procedure can be carried out:

- Fix the value  $x_{n+1}$  equal to  $x_{n+1}^\beta$  and solve the unconstrained maximal efficiency problem associated with the allocation of the remaining  $k - x_{n+1}^\beta$  servers to the groups  $1, 2, \dots, n$ . If the obtained solution  $x$  is feasible, that is,  $f_2(x) \leq \beta$  then the Pareto solution corresponding to  $\beta$  has been obtained. Otherwise fix the values of  $x_{n+1}$  and  $x_n$  equal to  $x_{n+1}^\beta$  and  $x_n^\beta$  respectively and repeat the procedure described above, *mutatis mutandis*. If the obtained solution is admissible, the Pareto solution has been found. Otherwise the described procedure is now applied to  $x_{n+1}$ ,  $x_n$ ,  $x_{n-1}$  and so on, until the Pareto solution has been found.

This type of procedure may be not very efficient for high values of  $n$  but it is efficient enough in many practical applications since usually  $n$  is not a high number. In any case this type of procedure is quite efficient for “traveling on  $\mathcal{N}$ ,” by enabling that a table of Pareto optimal solutions, with a certain pre-defined number  $N_p$  solutions, may be easily obtained. To show the effectiveness of the proposed algorithm for this purpose, we present the following Lemma.

**Lemma 11**

Let  $\beta$  and  $\beta'$  be values in the interval  $[\beta^{**}, \beta^*]$ . Then:

$$\beta' < \beta \implies \mathcal{A}(\beta) \subseteq \mathcal{A}(\beta').$$

**Proof:**

We need to prove that if a constraint is active on the optimal solution of Problem 3 for a certain value of  $\beta$  then this constraint is also active when a parameter  $\beta' < \beta$  is used. This is equivalent to prove that for all  $m \in \{1, 2, \dots, n, n+1\}$ :

$$m \in \mathcal{A}(\beta) \implies m \in \mathcal{A}(\beta').$$

If  $\beta' < \beta$ , from the monotonicity of the Erlang-B function,

$$x_m^\beta < x_m^{\beta'}. \quad (39)$$

In the problem corresponding to  $\beta$  the constraint  $m$  is  $x_m \geq x_m^\beta$  and in the problem for  $\beta'$ , the constraint is  $x_m \geq x_m^{\beta'}$ . Taking (39) into account and the convexity of function  $f_1$  the required result is obtained.  $\square$

From this Lemma it may be concluded that successive resolutions of Problem 3 with decreasing values of  $\beta$  makes that the fixed variables (corresponding to values  $x_m^\beta$ ) obtained for solving a certain problem will have to be fixed for obtaining the optimal solution to the following single criterion problem (with lower value of  $\beta$ ), possibly other variables having to be fixed.

Therefore for calculating a table of Pareto optimal solutions to Problem 1, we take successively smaller values  $\beta$  in the interval  $[\beta^{**}, \beta^*]$  and apply the general procedure explained above. If the step used for decrementing  $\beta$  is executed obviously the number of Pareto solutions which may be obtained increases and most of such solutions can be calculated by just solving a problem of maximal efficiency. Also note that the size of the solved problems decreases by one every time a variable is fixed, thence the numerical resolution becomes less heavy as the value of  $\beta$  comes nearer  $\beta^{**}$ .

Next the proposed algorithm for traveling on  $\mathcal{N}$  can be formalized. Following the described procedure, the algorithm enables the calculation of  $N_p$  Pareto solutions corresponding to decreasing values of  $\beta$  in the interval  $[\beta^{**}, \beta^*]$  by using a fixed decrement  $\delta = (\beta^* - \beta^{**})/(N_p + 1)$ . The extreme Pareto solutions  $x^*$  and  $x^{**}$  corresponding to  $\beta^*$  and  $\beta^{**}$  (maximal efficiency and maximal equity solutions respectively) are previously computed by using the method described in Sub-section 3.4 and Section 4.

**Algorithm 1** (“Traveling on  $\mathcal{N}$ ”)

*description*

*Input:* Number of groups  $N \geq 2$ ; parameter  $\kappa$ ; offered traffic vector  $A$ ; bounds  $\beta^*$  and  $\beta^{**}$ ; number of required Pareto optimal solutions  $N_p$  ;

```

1.   begin
2.    $\delta \leftarrow (\beta^* - \beta^{**}) / (N_p + 1);$ 
3.    $\beta \leftarrow \beta^*;$ 
4.    $N_\beta \leftarrow 1;$ 
5.   for  $j \leftarrow 1$  to  $N_p$  do
6.     begin
7.      $\beta \leftarrow \beta - \delta;$ 
8.      $S_\beta \leftarrow 0;$ 
9.     for  $i \leftarrow N$  to  $N - N_\beta$  do
10.      begin
11.       $X[i] \leftarrow XERL(A[i], \beta);$ 
12.       $S_\beta \leftarrow S_\beta + X[i];$ 
13.      end
14.      repeat
15.         $N_\ell \leftarrow N - N_\beta;$ 
16.         $\kappa_\beta \leftarrow \kappa - S_\beta;$ 
17.         $ALLOC(A, X, \kappa_\beta, N_\ell);$ 
18.         $M \leftarrow \max \{B(A[j], X[j]) : j = 1, \dots, N_\ell\};$ 
19.        if  $(M > \beta)$  then
20.          begin
21.             $N_\beta \leftarrow N_\beta + 1;$ 
22.             $I_g \leftarrow N - N_\beta + 1;$ 
23.             $X[I_g] \leftarrow XERL(A[I_g], \beta);$ 
24.             $S_\beta \leftarrow S_\beta + X[I_g];$ 
25.          end
26.        until  $(M \leq \beta);$ 
27.       $A_t \leftarrow \sum_{j=1}^N A[j] B(A[j], X[j]);$ 
28.      Pareto Point: Write  $(f_1 = A_t, f_2 = \beta)$  and the vector  $X$ ;
29.    end
30.  end;

```

*Output:* Table of Pareto optimal solutions;



In the description of the algorithm, *XERL* designates the subroutine which calculates the inverse of the Erlang-B function with respect to  $x$  (real number of servers) for given traffic offered and blocking probability. *ALLOC* represents the procedure that calculates the first  $N_l$  positions of the decision variable vector  $X$  corresponding to the maximal efficiency solution obtained by allocating a total capacity  $\kappa_\beta$  to the first  $N_l$  servers groups.

## 7 Some Computational Results

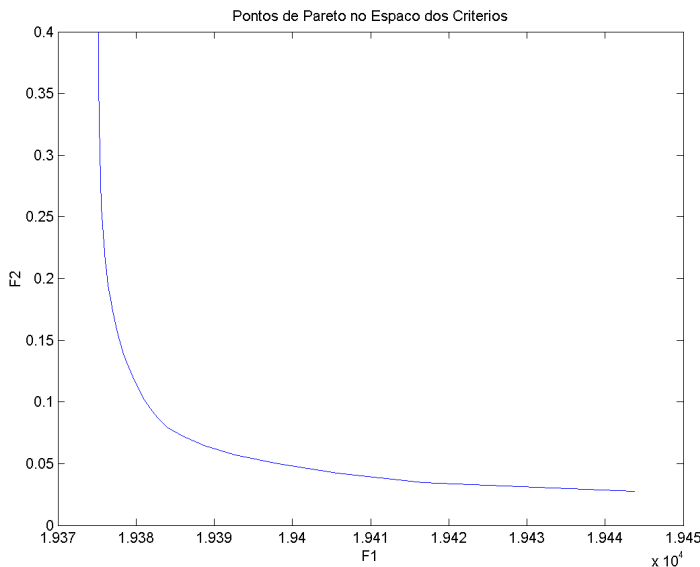
Some computational results are presented to illustrate the application of the proposed algorithm for  $n+1 = 25$  server groups and different values of the total number of servers  $\kappa$  and total offered traffic  $\alpha$ . The figures represent the Pareto efficient frontier in the objective function space. Examples of Pareto solutions are in table for the cases in Figures 1–3. The implementation of the algorithm was performed using Turbo C compiler version 2.1. on a PC (Intel Core2 dual E6400 processor running at 2.13 GHz and 2 GB RAM) using MS Windows XP.

Falta os seguintes valores:

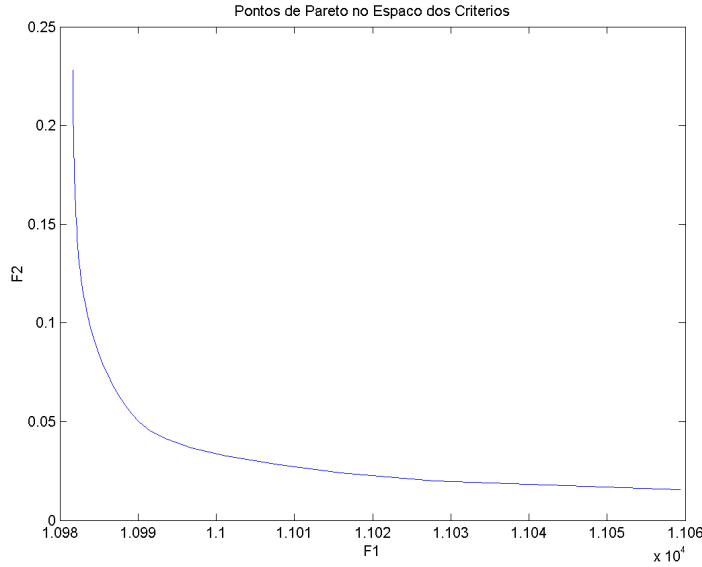
- Valor de  $\beta^{**}$  e de  $\beta^{ast}$ ;
- Valor de  $N_p$  ou o passo  $\delta$  no intervalo  $[\beta^{**}, \beta^*]$ ;
- valores de interesse  $x^*$ ,  $x^{**}$  e  $\mathbf{x}^*$  (ponto de distância de Tchebyshev pesada normalizada mínima ao ponto ideal);

•

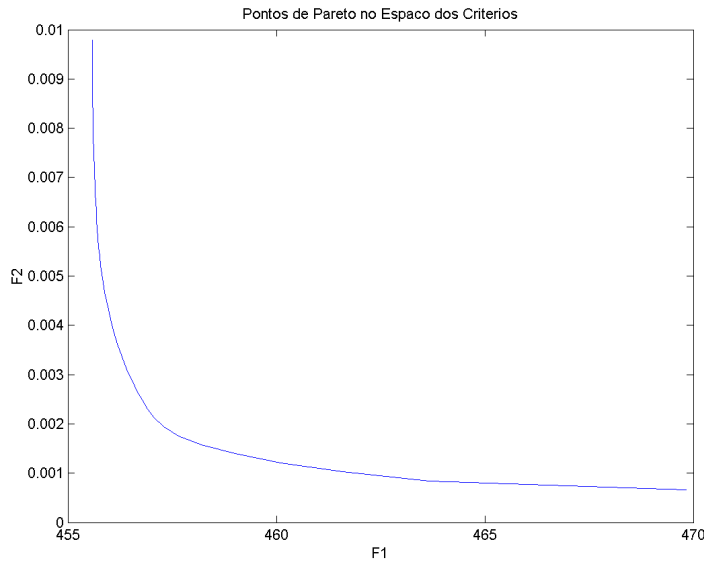
•



$$\mathbf{a} = \begin{bmatrix} 49\,743 \\ 49\,426 \\ 49\,036 \\ 47\,262 \\ 47\,204 \\ 39\,848 \\ 39\,146 \\ 36\,541 \\ 36\,114 \\ 35\,860 \\ 32\,707 \\ 32\,483 \\ 32\,361 \\ 32\,091 \\ 30\,664 \\ 23\,439 \\ 23\,311 \\ 23\,188 \\ 20\,805 \\ 18\,125 \\ 3\,739 \\ 3\,196 \\ 2\,912 \\ 381 \\ 158 \end{bmatrix}, \text{ with: } \begin{cases} \#25 \text{ groups} \\ \alpha = 709\,740 \\ \kappa = 691\,100 \end{cases}$$



$$\mathbf{a} = \begin{bmatrix} 49\,743 \\ 49\,426 \\ 49\,036 \\ 47\,262 \\ 47\,204 \\ 39\,848 \\ 39\,146 \\ 36\,541 \\ 36\,114 \\ 35\,860 \\ 32\,707 \\ 32\,483 \\ 32\,361 \\ 32\,091 \\ 30\,664 \\ 23\,439 \\ 23\,311 \\ 23\,188 \\ 20\,805 \\ 18\,125 \\ 3\,739 \\ 3\,196 \\ 2\,912 \\ 381 \\ 158 \end{bmatrix}, \text{ with: } \begin{cases} \#25 \text{ groups} \\ \alpha = 709\,740 \\ \kappa = 700\,000 \end{cases}$$



$$\mathbf{a} = \begin{bmatrix} 49\,743 \\ 49\,426 \\ 49\,036 \\ 47\,262 \\ 47\,204 \\ 39\,848 \\ 39\,146 \\ 36\,541 \\ 36\,114 \\ 35\,860 \\ 32\,707 \\ 32\,483 \\ 32\,361 \\ 32\,091 \\ 30\,664 \\ 23\,439 \\ 23\,311 \\ 23\,188 \\ 20\,805 \\ 18\,125 \\ 3\,739 \\ 3\,196 \\ 2\,912 \\ 381 \\ 158 \end{bmatrix}, \text{ with: } \begin{cases} \#25 \text{ groups} \\ \alpha = 709\,740 \\ \kappa = 716\,000 \end{cases}$$

The CPU times in these experiments were of the order of ...??... (seconds). These results illustrate that the proposed algorithm is very effective in generating the Pareto frontier, in the envisaged type of applications of the bicriteria model.

## 8 Conclusions

The design of telecommunication networks is a highly complex decision problem involving the extensive use of decompositions of the associated large-scale optimization problems. Also the stochastic nature of the offered demand often requires the consideration of stochastic service

system models in association with optimization problems. Furthermore the mathematical formulations of many design problem become more realistic and powerful if various relevant criteria are represented explicitly rather than aggregated *a priori* in a single function to be optimized. This justifies the increasing interest and potential advantages in using multicriteria in this area which enable a mathematically consistent treatment of the trade-off between multiple, conflicting criteria. Following this methodological trend we approached a capacity allocation optimization problem on a classical stochastic service system, the Erlang-B  $M/M/x$  — loss system, by proposing a bicriteria formulation.

The problem is focused on the allocation of a given total of  $\kappa > 0$  servers to a number of groups of servers capable of carrying certain offered traffic processes assumed as Poissonian in nature. Two main objectives were present in this formulation. Firstly a criterion of equity in the grade of service, measured by the call blocking probabilities, establishing that the absolute difference between the blocking probabilities experienced by the calls in the different service groups must be as small as possible. Secondly a criterion of system economic performance, to be optimized, was introduced. This criterion implies that the total traffic carried by the system should be maximized. Relevant mathematical results characterizing the two objective functions and the set  $\mathcal{N}$  of the non-dominated solutions, were presented. An algorithm for traveling on  $\mathcal{N}$  based on the resolution of monocriteria convex problems, using a Newton-Raphson method, was also proposed. In each iteration the first two derivatives of the Erlang-B function in the number of circuits (a difficult numerical problem) were calculated using a method earlier proposed. Some computational results obtained with the algorithm were also presented which illustrate the effectiveness of the proposed approach.

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## Appendix A Application of the Newton Method

The application of Newton method to the special class of nonlinear systems which appear in this paper is now discussed. Let us introduce the function  $\Phi \in \mathcal{C}^2(\mathcal{D})$ :

$$\begin{aligned} \Phi : \mathcal{D} \subset \mathbb{R}^p &\longrightarrow \mathbb{R}^p \\ \mathbf{x} &\longmapsto \Phi(\mathbf{x}), \end{aligned}$$

such that  $\Phi(\mathbf{x}) = \mathbf{0}$  have unique solution on  $\mathcal{D}$ .

If  $\Phi'(\mathbf{x})$  denote the Jacobian matrix (evaluated at  $\mathbf{x}$ ) of the function  $\Phi$ , then, given an iterate  $\mathbf{x}$ , Newton's method generates the next iterate  $\mathbf{x}^+$  by solving the linear system:

$$\Phi'(\mathbf{x}) \mathbf{y} = -\Phi(\mathbf{x}), \quad (40)$$

for the correction vector  $\mathbf{y}$ , and setting  $\mathbf{x}^+ = \mathbf{x} + \mathbf{y}$ .

The special class of nonlinear systems  $\Phi(\mathbf{x}) = \mathbf{0}$  which appear in this paper is characterized by the following conditions:

$$\begin{aligned} \Phi'(\mathbf{x}) &= g(\mathbf{x}) [E + \text{diag}(\mathbf{w})], \\ \mathbf{w} &\in \mathbb{R}_+^n, \\ g(\mathbf{x}) &\neq 0, \end{aligned}$$

where  $E$  is a matrix having all entries equal to 1. In this special case, the linear system (40) is equivalent to:

$$[E + \text{diag}(\mathbf{w})] \mathbf{y} = \mathbf{b}, \quad \text{where} \quad \mathbf{b} = -\frac{1}{g(\mathbf{x})} \Phi(\mathbf{x}).$$

Note that this linear system is completely specified by the vectors  $\mathbf{w}$  and  $\mathbf{b}$ . Next lemma shows that such class of Cramer linear systems are very easy to solve. Clearly the number of arithmetic operations involved in the process is proportional to the order of the system, as for diagonal systems.

### Lemma 12

*Denoting by  $E \in \mathbb{R}^{p \times p}$  the matrix having all its entries equal to 1, if  $\mathbf{w} \in \mathbb{R}_+^p$  and  $\mathbf{b} \in \mathbb{R}^p$ , the solution of the linear system:*

$$[E + \text{diag}(\mathbf{w})] \mathbf{y} = \mathbf{b} \quad (41)$$

*is given by*

$$y_j = \frac{b_j - \sigma}{w_j}, \quad j = 1(1)p, \quad \text{where} \quad \sigma = \frac{\sum_{i=1}^p (b_i/w_i)}{1 + \sum_{i=1}^p (1/w_i)} = \sum_{i=1}^p y_i.$$

### Proof:

The system may be written as  $M \mathbf{y} = \mathbf{b}$ , where  $M = E + \text{diag}(\mathbf{w})$ . Introducing the notation,

$$\begin{aligned} \mathbf{e} &= [1 \ 1 \ 1 \ \cdots \ 1]^T \in \mathbb{R}^p, \\ D &= \text{diag}(\mathbf{w}), \end{aligned}$$

we have,

$$M \mathbf{y} = \mathbf{b} \iff E \mathbf{y} + D \mathbf{y} = \mathbf{b} \iff D \mathbf{y} + \left( \sum_{i=1}^p y_i \right) \mathbf{e} = \mathbf{b}.$$

Defining the variable  $y_{p+1} = \sum_{i=1}^p y_i$ , system  $M \mathbf{y} = \mathbf{b}$  is equivalent to

$$\begin{cases} D \mathbf{y} + y_{p+1} \mathbf{e} = \mathbf{b} \\ -\mathbf{e}^T \mathbf{y} + y_{p+1} = 0 \end{cases} \iff \begin{bmatrix} D & \mathbf{e} \\ -\mathbf{e}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ y_{p+1} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}.$$

Using Gaussian elimination, we obtain:

$$\begin{bmatrix} D & \mathbf{e} \\ \mathbf{0}^T & 1 + \sum_{i=1}^n (1/w_i) \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \sum_{i=1}^n (b_i/w_i) \end{bmatrix}.$$

Using inverse substitution, the result follows with  $\sigma = y_{p+1}$ .  $\square$

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